Bayesian Games and the Smoothness Framework

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Abstract

We consider a general class of Bayesian Games where each players utility depends on his type (possibly multidimensional) and on the strategy profile and where players' types are distributed independently. We show that if their full information version for any fixed instance of the type profile is a smooth game then the Price of Anarchy bound implied by the smoothness property, carries over to the Bayes-Nash Price of Anarchy. We show how some proofs from the literature (item bidding auctions, greedy auctions) can be cast as smoothness proofs or be simplified using smoothness. For first price item bidding with fractionally subadditive bidders we actually manage to improve by much the existing result [6] from 4 to $\frac{e}{e-1} \approx 1.58$. This also shows a very interesting separation between first and second price item bidding since second price item bidding has PoA at least 2 even under complete information. For a larger class of Bayesian Games where the strategy space of a player also changes with his type we are able to show that a slightly stronger definition of smoothness also implies a Bayes-Nash PoA bound. We show how weighted congestion games actually satisfy this stronger definition of smoothness. This allows us to show that the inefficiency bounds of weighted congestion games known in the literature carry over to incomplete versions where the weights of the players are private information. We also show how an incomplete version of a natural class of monotone valid utility games, called effort market games are universally $(1, 1)$-smooth. Hence, we show that incomplete versions of effort market games where the abilities of the players and their budgets are private information has Bayes-Nash PoA at most 2.

1 Introduction

In our information era, with the advent of electronic markets, most systems have grown so large scale that central coordination has become infeasible. In addition players are less and less informed of the actual game they are playing and the type of players they are competing against. Coordinating the players is too costly and assuming that the players know all the parameters of the game is too simplistic. Such a realization renders mandatory the study of efficiency in non-cooperative games of incomplete information.

Ever since the introduction of the concept of the Price of Anarchy, a large part of the algorithmic game theory literature has studied the effects of selfishness in the efficiency of a system. In a unifying paper, Roughgarden [11], gave a general technique, called smoothness, for proving inefficiency results in games and portrayed how many of the results in the literature can be cast in his framework. In addition, he showed that such types of inefficiency proofs extend directly to almost every reasonable non-cooperative solution concept, such as pure nash equilibria, mixed nash equilibria, correlated equilibria and coarse-correlated equilibria.

However, such a unification holds only under the strong assumption of complete information [1]: players know every parameter of the game and no player has private information. Such an assumption is pretty strong and if we want models that could capture realistic environments we need to cope with games were players have incomplete information.

In this work we manage to show how to extend this unification to a significant class of games of incomplete information: basically we show that if a complete information game is smooth then the inefficiency bound given by the smoothness argument carries over to incomplete information versions of the game where players have private parameters and each player's utility depends on his parameter and the actions of the rest of the players. Hence, we manage to unify Price of Anarchy with Bayesian Price of Anarchy analysis for a large class of games.

[1] Recently we became aware that an independent work by Roughgarden with some overlapping results on extending smoothness to incomplete information games has been under submission to a conference since February 6, 2012.

Many of the games studied in the literature such as Weighted Congestion Games have been shown to be tight games: the best efficiency bound provable with a smoothness argument is the best bound possible. For such games our analysis shows that the bayesian price of anarchy bound is also tight, since complete information is a special case of incomplete information and hence lower bounds on inefficiency carry over to the incomplete information model. Hence, this immediately implies that the efficiency guarrantees for a tight class of games don't depend on the information pocessed by the players: having more information doesn't imply better efficiency and having less information doen't imply worse efficiency.

Our approach also manages to put several Bayesian Price of Anarchy results that exist in the literature under the smoothness framework. Specifically we show how our main theorem can be applied to get an improved result for first price item-bidding with subadditive bidders and how to get a much simpler proof of good approximation guarrantees of the greedy mechanisms introduced by Lucier and Borodin [8].

1.1 Related Work

There has been a long line of research on quantifying inefficiency of equilibria starting from [7] who introduced the notion of the price of anarchy. A recent work by Roughgarden [11] managed to unify several of these results under a proof framework called smoothness and also showed that such inefficiency proofs also carry over to inefficiency of coarse correlated equilibria. Moreover, he showed that such techniques give tight results for the well-studied class of congestion games. Later, Bhawalkar et al. [2] also showed that it produces tight results for the larger class of weighted congestion games. Another recent work by Schoppman and Roughgarden [12] copes with games with continuous strategy spaces and shows how the smoothness framework should be adapted for such games to produce tighter results. The introduce the new notion of local smoothness for such games and showed that if an inefficiency upper bound proof lies in this framework then it also carries over to correlated equilibria.

There have also been several works on quantifying the inefficiency of incomplete information games, mainly in the context of auctions. A series of papers by Paes Leme and Tardos [10], Lucier and Paes Leme [9] and Caragiannis et al [4] studied the ineffficiency of Bayes-Nash equilibria of the generalized second price auction. Lucier and Borodin studied Bayes-Nash Equilibria of non-truthful auctions that are based on greedy allocation algorithms [8]. A series of three papers, Christodoulou, Kovacs and Schapira [5], Bhawalkar and Roughgarden [3] and Hassidim, Kaplan, Mansour, Nisan [6], studied the inefficiency of Bayes-Nash equilibria of non-truthful combinatorial auctions that are based on running simultaneous separate item auctions for each item. However, many of the results in this line of work where specific to the context and a unifying framework doesn't exist. Lucier and Paes Leme [9] introduced the concept of semi-smoothness and showed that their proof for the inefficiency of the generalized second price auction falls into this category. However, semi-smoothness is a much more restrictive notion of smoothness than just requiring that every complete information instance of the game to be smooth.

1.2 Model and Notation

We consider the following model of Bayesian Games: Each player has a type space T_i and a probability distribution D_i defined on T_i. The distributions D_i are independent, the types T_i are disjoint and we denote with $D = \times_i D_i$. Each player has a set of actions A_i and let $A = \times_i A_i$. The utility of a player is a function $u_i : T_i \times A \to \mathbb{R}$. The strategy of each player is a function $s_i : T_i \to A_i$. At times we will use the notation $s(t) = (s_i(t_i))_{i\in N}$ to denote the vector of actions given a type profile t and $s_{-i}(t_{-i}) = (s_j(t_j))_{j\neq i}$ to denote the vector of actions for all players except i. We could also define cost minimization games where each player has a cost $c_i : T_i \times A \to \mathbb{R}$. All of our results hold for both utility maximization and cost minimization games.

The two basic assumptions that we made in the class of Bayesian Games that we examine is that a players utility is affected by the other players' types only implicitly through their actions and not directly from their types and that the players' types are distributed independently. The above class of games is general enough and we portray several nice examples of such Bayesian Games. An interesting future direction is to try and relax any of these two assumptions or show that smoothness is not sufficient to prove bounds without these assumptions.

As our solution concept we will use the most dominant solution concept in incomplete information games, the Bayes-Nash Equilibrium (BNE). Our results hold for mixed Bayes-Nash Equilibria too, but for

simplicity of presentation we are going to focus on Bayes-Nash Equilibria in pure strategies. A Bayes-Nash Equilibrium is a strategy profile such that each player maximizes his expected utility conditional on his private information:

$$\forall a \in A_i : \mathbb{E}_{t_{-i}|t_i}[u_i^{t_i}(s(t))] \geq \mathbb{E}_{t_{-i}|t_i}[u_i^{t_i}(a, s_{-i}(t_{-i})]$$

Given a strategy profile s the social welfare of the game is defined as the expected sum of player utilities:

$$SW(s) = \mathbb{E}_t[SW^t(s(t))] = \mathbb{E}_t[\sum_i u_i^{t_i}(s(t))]$$

In addition given a type profile t we denote with $\textsc{Opt}(t)$ the action profile that achieves maximum social welfare for type profile t: $\textsc{Opt}(t) = \arg\max_{a \in A} SW^t(a)$.

As our measure of inefficiency we will use the *Bayes-Nash Price of Anarchy* which is defined as the ratio of the expected optimal social welfare over the expected social welfare achieved at the worst Bayes-Nash Equilibrium:

$$\sup_{s \text{ is } BNE} \frac{\mathbb{E}_t[SW(\textsc{Opt}(t))]}{\mathbb{E}_t[SW(s(t))]}$$

2 Constant Strategy Space Bayesian Games and Smoothness

In this section we give a general theorem on how one can derive Bayes-Nash Price of Anarchy results using the smoothness framework introduced in [11]. We first give a definition of smoothness suitable for incomplete information games. Our definition just states that the game is smooth in the sense of [11] for each instantiation of the type profile.

Definition 1 *A bayesian utility maximization game is said to be (λ, μ)-smooth if for any $t \in T$ and for any pair of action profiles $a, a' \in A$:*

$$\sum_i u_i^{t_i}(a_i', a_{-i}) \geq \lambda SW^t(a') - \mu SW^t(a)$$

For the case of complete information games (i.e. the set of possible type profiles is a singleton) Roughgarden [11] showed that the efficiency achieved by any coarse-correlated equilibrium (superset of pure nash, mixed nash and correlated equilibrium) is at least a $\lambda/(1+\mu)$ fraction of the optimal social welfare. In other words the price of anarchy of any of these solution concepts is at most $(1+\mu)/\lambda$. Our main results shows that the latter also extends to Bayes-Nash Equilibria for the case of incomplete information.

Theorem 2 (Main Theorem) *If a Bayesian Game is (λ, μ)-smooth then it has Bayes-Nash Price of Anarchy at most $\frac{1+\mu}{\lambda}$.*

Proof. Let $\textsc{Opt}(t) = (\textsc{Opt}_i(t))_{i \in N}$ be the action profile that maximizes social welfare given a type profile t. Suppose that player i with type t_i switches to playing $\textsc{Opt}_i(t_i, w_{-i})$ for some type profile w_{-i}. Let s be a Bayes-Nash Equilibrium. Then we have:

$$\mathbb{E}_{t_{-i}}[u_i^{t_i}(s(t))] \;\geq\; \mathbb{E}_{t_{-i}}[u_i^{t_i}(\textsc{Opt}_i(t_i, w_{-i}), s_{-i}(t_{-i}))]$$

Taking expectation over t_i and over all possible type profiles w_{-i} we get:

$$\begin{aligned}
\mathbb{E}_t[u_i^{t_i}(s(t))] &\geq \mathbb{E}_{w_{-i}}\mathbb{E}_{t_i}\mathbb{E}_{t_{-i}}[u_i^{t_i}(\textsc{Opt}_i(t_i, w_{-i}), s_{-i}(t_{-i}))] \\
&= \mathbb{E}_{w_{-i}}\mathbb{E}_{w_i}\mathbb{E}_{t_{-i}}[u_i^{w_i}(\textsc{Opt}_i(w_i, w_{-i}), s_{-i}(t_{-i}))] \\
&= \mathbb{E}_{t_i}\mathbb{E}_{w_{-i}}\mathbb{E}_{w_i}\mathbb{E}_{t_{-i}}[u_i^{w_i}(\textsc{Opt}_i(w_i, w_{-i}), s_{-i}(t_{-i}))] \\
&= \mathbb{E}_t\mathbb{E}_w[u_i^{w_i}(\textsc{Opt}_i(w), s_{-i}(t_{-i}))]
\end{aligned}$$

Adding the above inequality for all players and using the smoothness property we get:

$$\begin{aligned}
\mathbb{E}_t[SW^t(s(t))] = \sum_i \mathbb{E}_t[u_i^{t_i}(s(t))] &\geq \sum_i \mathbb{E}_t\mathbb{E}_w[u_i^{w_i}(\textsc{Opt}_i(w), s_{-i}(t_{-i}))] \\
&= \mathbb{E}_t\mathbb{E}_w[\sum_i u_i^{w_i}(\textsc{Opt}_i(w), s_{-i}(t_{-i}))] \\
&\geq \mathbb{E}_t\mathbb{E}_w[\lambda SW^w(\textsc{Opt}(w)) - \mu SW^w(s(t))] \\
&= \lambda \mathbb{E}_w[SW^w(\textsc{Opt}(w))] - \mu \sum_i \mathbb{E}_t\mathbb{E}_w[u_i^{w_i}(s_i(t_i), s_{-i}(t_{-i}))] \qquad (1)
\end{aligned}$$

If in the pre-last line we had $SW^t(s(t))$ then we would directly get our result. However, the fact that there is this misalignment we need more work. In fact we are going to prove that:

$$\mathbb{E}_t\mathbb{E}_w[SW^w(s(t))] \leq \mathbb{E}_t[SW^t(s(t))] \qquad (2)$$

To achieve this we are going to use again the Bayes-Nash Equilibrium definition. But now we are going to use it in the following sense: no player i of some type w_i wants to deviate to playing as if he was some other type t_i. This translates to:

$$\mathbb{E}_{t_{-i}}[u_i^{w_i}(s_i(t_i), s_{-i}(t_{-i})] \leq \mathbb{E}_{t_{-i}}[u_i^{w_i}(s_i(w_i), s_{-i}(t_{-i})]$$

Taking expectation over t_i and w we have:

$$\begin{aligned}
\mathbb{E}_w\mathbb{E}_t[u_i^{w_i}(s_i(t_i), s_{-i}(t_{-i})] &\leq \mathbb{E}_{w_i}\mathbb{E}_{w_{-i}}\mathbb{E}_{t_i}\mathbb{E}_{t_{-i}}[u_i^{w_i}(s_i(w_i), s_{-i}(t_{-i})] \\
&= \mathbb{E}_{w_i}\mathbb{E}_{t_{-i}}[u_i^{w_i}(s_i(w_i), s_{-i}(t_{-i})] \\
&= \mathbb{E}_{t_i}\mathbb{E}_{t_{-i}}[u_i^{t_i}(s_i(t_i), s_{-i}(t_{-i})] \\
&= \mathbb{E}_t[u_i^{t_i}(s(t))]
\end{aligned}$$

Summing over all players gives us inequality 2. Now combining inequality 2 with inequality 1 we get:

$$\mathbb{E}_t[SW^t(s(t))] \geq \lambda \mathbb{E}_w[SW^w(\textsc{Opt}(w))] - \mu \mathbb{E}_t[SW^t(s(t))] \qquad (3)$$

which gives the theorem. ■

In fact it is easy to see that our above analysis also works for a more relaxed version of smoothness similar to the variant introduced in [9]:

Definition 3 *A Bayesian utility maximization game is said to be (λ, μ)-smooth if for any $t \in T$ and $a \in A$, there exists a strategy profile $a'(t)$ such that:*

$$\sum_i u_i^{t_i}(a_i'(t), a_{-i}) \geq \lambda SW(\textsc{Opt}(t)) - \mu SW^t(a)$$

In fact our main proof holds even for a slightly more relaxed smoothness property that will prove useful in auction settings. Our main theorem works for the following notion of smoothness under the condition that the utilities of players at equilibrium are non-negative.

Definition 4 *A Bayesian utility maximization game is said to be (λ, μ)-smooth if for any $t \in T$ and $a \in A$, there exists a strategy profile $a'(t)$ such that:*

$$\sum_i u_i^{t_i}(a_i'(t), a_{-i}) \geq \lambda SW(\textsc{Opt}(t)) - \mu \sum_{i \in K \subseteq [n]} u_i^t(a)$$

where K is some fixed subset of the players independent of the type profile t and bid profile b.

The reason why the latter definition is useful is that some auction environments might fail to be smooth with the first definition mainly due to the fact that the utility of players might be non-negative in expectation at equilibrium but can certainly be negative if we consider an arbitrary bid profile b that is not in equilibrium. Hence, the latter helps in settings where at equilibrium utilities are certainly non-negative (individual rationality) but there are strategy profiles at which a player might be getting negative utility.

3 Item-Bidding Auctions

In this section we consider the Item-Bidding Auctions studied in Christodoulou et al [5], Bhawalkar and Roughgarden [3] and Hassidim et al. [6].

We first prove a smoothness result for First Price Item-Bidding Auctions for fractionally subadditive bidders. Fractionally subadditive bidders are subcase of additive bidders and a generalization of submodular bidders. Our results imply a big improvement in existing results. Specifically Hassidim et al show that for fractionally subadditive bidders the Bayes-Nash Price of Anarchy is at most 4 (and at most 4β for β-fractionally subadditive. We show that it is at most $\frac{e}{e-1} \approx 1.58$ (and at most $\frac{e}{e-1}\beta$ correspondingly).

Thus our result gives the same guarantees for first price item bidding auctions as those existing for second price item bidding auctions (e.g. Christodoulou et al [5], Bhawalkar et al [3]).

Theorem 5 *First Price Item-Bidding Auctions are $(\frac{1}{2}, 0)$-semi-smooth for fractionally subadditive bidders.*

Proof. Consider a valuation profile v and a bid profile b. Let $\textsc{Opt}(v)$ be the optimal allocation for that type profile. Let $\textsc{Opt}_i(v)$ be the set that player i gets at $\textsc{Opt}(v)$. Let $a = (a_1, \dots, a_m)$ be the maximizing additive valuation for player i for $\textsc{Opt}_i(v)$, i.e. $v_i(\textsc{Opt}_i(v)) = \sum_{j \in \textsc{Opt}_i(v)} a_j$ and $\forall S \neq \textsc{Opt}_i(v) : v_i(S) \geq \sum_{j \in S} a_j$ (Such an a exists by the definition of fractionally subadditive valuations). Suppose that player i switches to bidding $a_j/2$ for each $j \in \textsc{Opt}_i(v)$ and 0 everywhere else. Denote with $b_i'(v)$ such a deviation. Let X_i be the items that he wins after the deviation. This means that for all $j \in \textsc{Opt}_i(v) - X_i : p_j(b) \geq a_j/2$ and for all $j \in X_i$ player i pays exactly $a_j/2$. Thus we have:

$$\begin{aligned}
u_i(b_i'(v), b_{-i}) &\geq v_i(X_i) - \sum_{j \in X_i} \frac{a_j}{2} \geq \sum_{j \in X_i} a_j - \sum_{j \in X_i} \frac{a_j}{2} = \sum_{j \in X_i} \frac{a_j}{2} \\
&\geq \sum_{j \in X_i} \frac{a_j}{2} + \sum_{j \in \textsc{Opt}_i(v) - X_i} \frac{a_j}{2} - p_j(b) \\
&\geq \sum_{j \in \textsc{Opt}_i(v)} \frac{a_j}{2} - \sum_{j \in \textsc{Opt}_i(v) - X_i} p_j(b) \\
&\geq \sum_{j \in \textsc{Opt}_i(v)} \frac{a_j}{2} - \sum_{j \in \textsc{Opt}_i(v)} p_j(b) \\
&= \frac{v_i(\textsc{Opt}_i(v))}{2} - \sum_{j \in \textsc{Opt}_i(v)} p_j(b)
\end{aligned}$$

Now summing over all players the second term on the right hand side will become the sum of prices over all items, since the sets $\textsc{Opt}_i(v)$ are disjoint. Thus we get:

$$\sum_i u_i^{v_i}(b_i'(v), b_{-i}) + \sum_{j \in [m]} p_j(b) \geq \frac{\sum_i v_i(\textsc{Opt}_i(v))}{2} = \frac{1}{2} SW^v(\textsc{Opt}(v)) \tag{4}$$

Now the above inequality gives us the smoothness property. To completely fit it in our smoothness model, we should also view the seller as a player with only one strategy and only one type and whose utility is the sum of payments. His optimal deviation is then the trivial of doing nothing. Then the left hand side of the above inequality is the sum of the utilities of all the players (including the seller) had each of them unilaterally deviated to their optimal strategy. ■

In fact considering randomized deviations, similar to that of [9] we are able to prove a much tighter bound of $\frac{e}{e-1} \approx 1.58$ on the Price of Anarchy by showing that the above game is actually $(1 - \frac{1}{e}, 0)$-semi-smooth.

Theorem 6 *First Price Item-Bidding Auctions are $(1 - \frac{1}{e}, 0)$-semi-smooth for fractionally subadditive bidders.*

Proof. Consider a valuation profile v and a bid profile b. Let $\textsc{Opt}(v)$ be the optimal allocation for that type profile. Let $\textsc{Opt}_i(v)$ be the set that player i gets at $\textsc{Opt}(v)$. Let $a = (a_1, \dots, a_m)$ be the maximizing additive

valuation for player i for $\textsc{Opt}_i(v)$, i.e. $v_i(\textsc{Opt}_i(v)) = \sum_{j\in\textsc{Opt}_i(v)} a_j$ and $\forall S \neq \textsc{Opt}_i(v) : v_i(S) \geq \sum_{j\in S} a_j$ (Such an a exists by the definition of fractionally subadditive valuations). Suppose that player i switches to bidding a randomized bid with probability density function $f(b) = \frac{1}{a_j - b}$ for $b \in [0, a_j(1-\frac{1}{e})]$ for each $j \in \textsc{Opt}_i(v)$ and 0 everywhere else. Randomization is independent for each j. Denote with $\tilde{B}_i(v)$ such a randomized deviation and $\tilde{b}_i$ a random draw from $\tilde{B}_i(v)$. Let $X_i(b_i)$ be the random variable that denotes the items that he wins after the deviation. Thus we have:

$$\begin{aligned}
u_i(\tilde{B}_i(v), b_{-i}) &\geq \mathbb{E}_{\tilde{b}_i\sim\tilde{B}_i(v)}[v_i(X_i(\tilde{b}_i)) - \sum_{j\in X_i(\tilde{b}_i)} \tilde{b}_{ij}] \\
&\geq \mathbb{E}_{\tilde{b}_i\sim\tilde{B}_i(v)}[\sum_{j\in X_i(\tilde{b}_i)} a_j - \sum_{j\in X_i(\tilde{b}_i)} \tilde{b}_{ij}] \\
&\geq \mathbb{E}_{\tilde{b}_i\sim\tilde{B}_i(v)}[\sum_{j\in\textsc{Opt}_i(v)} (a_j - \tilde{b}_{ij})\mathbb{1}\{\tilde{b}_{ij} \geq p_j(b)\}] \\
&= \sum_{j\in\textsc{Opt}_i(v)} \mathbb{E}_{\tilde{b}_i\sim\tilde{B}_i(v)}[(a_j - \tilde{b}_{ij})\mathbb{1}\{\tilde{b}_{ij} \geq p_j(b)\}] \\
&= \sum_{j\in\textsc{Opt}_i(v)} \int_{p_j(b)}^{a_j(1-\frac{1}{e})} (a_j - t)\frac{1}{a_j - t}dt \\
&= \sum_{j\in\textsc{Opt}_i(v)} a_j\left(1 - \frac{1}{e}\right) - p_j(b) \\
&= \left(1 - \frac{1}{e}\right) v_i(\textsc{Opt}_i(v)) - \sum_{j\in\textsc{Opt}_i(v)} p_j(b)
\end{aligned}$$

Now summing over all players the second term on the right hand side will become the sum of prices over all items, since the sets $\textsc{Opt}_i(v)$ are disjoint. Thus we get:

$$\sum_i u_i^{v_i}(b_i'(v), b_{-i}) + \sum_{j\in[m]} p_j(b) \geq \left(1 - \frac{1}{e}\right)\sum_i v_i(\textsc{Opt}_i(v)) = \left(1 - \frac{1}{e}\right) SW^v(\textsc{Opt}(v)) \tag{5}$$

■

Similarly one can also show that for β-fractionally subadditive bidders the First Price Item-Bidding Auction is $(\frac{1}{\beta}\left(1 - \frac{1}{e}\right), 0)$-semi-smooth.

Corollary 7 *First Price Item-Bidding Auctions with independent β-fractionally subadditive bidders have Bayes-Nash Price of Anarchy at most $\frac{e}{e-1}\beta$.*

It is known (see [3]) that subadditive bidders are $\ln m$-fractionally subadditive. Thus the latter gives a price of anarchy of $\frac{e}{e-1}\ln m$ for subadditive bidders.

The latter result creates an interesting separation between first-price and second price auctions even in the incomplete information case. For the complete information case it is known that second price item bidding has price of anarchy at least 2 [5], whilst pure nash equilibria of first price auctions are always optimal (when a pure nash exists) [6]. The above bound states that such a separation even in the incomplete information case, since second price has price of anarchy at least 2 whilst first price auctions have price of anarchy at most ≈ 1.58.

4 Greedy First Price Auctions

In this section we consider the greedy first price auctions introduced by Lucier and Borodin [8]. In a greedy auction setting there are n bidders and m items. Each bidder i have some private combinatorial valuation v_i on the items that is drawn from some commonly known distribution. The strategies of the players is to submit a valuation profile b_i that outputs a value $b_i(S)$ for each set of items S. An allocation is a vector

$A = (A_1, \dots, A_n)$ that allocates a set A_i for each player i. We assume that there is a predefined subspace of feasible allocations. The above setting is a generalization of the combinatorial auction setting since we don't assume that the allocations A_i must be disjoint, i.e. the same item can potentially be allocated to more than one players.

A mechanism $\mathcal{M}(b) = (\mathcal{A}(b), p(b))$ takes as input a bid profile b and outputs a feasible allocation $\mathcal{A}(b)$ and a vector of prices $p(b)$ that each player has to pay. A mechanism is said to be greedy if the allocation output by the mechanism is the outcome of a greedy algorithm as we explain. Given bid profile b a greedy algorithm is defined as follows: Let $r : [n] \times 2^{[m]} \times \mathbb{R} \to \mathbb{R}$ be a priority function such that $r(i, S, v)$ is the priority of allocating set S to player i when $b_i(S) = v$. Then the greedy algorithm is as follows: Pick allocation (i, A_i) that maximizes $r(i, S, b_i(S))$ over all currently feasible allocations (i, S) and allocate set A_i to player i. Then remove player i. A greedy algorithm is c-approximate if for any bid profile b it returns an allocation that is at least a c-fraction of the maximum possible allocation given valuation profile b. Moreover, the mechanism is first price if the payments that the mechanism outputs is $p_i(b) = b_i(A_i)$.

Given a type profile v and a bid strategy profile b the social welfare is as always the sum of the players' utilities (including the auctioneer as a player). This boils down to being the value of the allocation $\sum_i v_i(A_i)$ since payments cancel out.

The game defined by a greedy mechanism is a Separable Bayesian Game and in the theorem that follows we are able to prove that it is also $(\frac{1}{2}, c-1)$ smooth. This leads to a price of anarchy of $2c$. This is not an improvement to the existing result of $c + O(\log(c))$ by Lucier and Borodin, but the analysis is much simpler and the difference in the two bounds is not big.

Theorem 8 *Any Greedy First Price Mechanism based on a c-approximate greedy algorithm defines a Bayesian Game that is $(\frac{1}{2}, c-1)$-smooth.*

Proof. Given the mechanism $\mathcal{M}$ we define as $\theta_i(S, b_{-i})$ as the critical value that player i has to bid on set S such that he is allocated set S given the bid profiles of the rest of the players.

In our proof we will use a very nice fact about c-approximate greedy mechanisms that was proved by Lucier and Borodin. For any c-approximate greedy mechanism and any feasible allocation A' it must hold that $\sum_i \theta_i(A'_i, b_{-i}) \leq c \sum_i b_i(A_i) = c \sum_i p_i(b)$.

Now consider a valuation profile v and bid profile b. Let $\textsc{Opt}_i(v)$ be the set allocated to player i in the optimal allocation for v. Let $b'_i(v)$ be the following bid strategy for player i: he bids $v_i(\textsc{Opt}_i(v))/2$ single-mindedly on $\textsc{Opt}_i(v)$, i.e. $\forall S \neq \textsc{Opt}_i(v) : b_i(S) = 0$ and $b_i(\textsc{Opt}_i(v)) = \frac{v_i(\textsc{Opt}_i(v))}{2}$. There are two cases: either he gets allocated his optimal item in which case his $u_i(b'_i(v), b_{-i}) = \frac{v_i(\textsc{Opt}_i(v))}{2}$ or he doesn't in which case: $\theta_i(\textsc{Opt}_i(v), b_{-i}) \geq \frac{v_i(\textsc{Opt}_i(v))}{2}$. Therefore, we get:

$$u_i(b'_i(v), b_{-i}) \geq \frac{v_i(\textsc{Opt}_i(v))}{2} - \theta_i(\textsc{Opt}_i(v), b_{-i}) \tag{6}$$

Summing over all players and using the upper bound on critical price proved by Lucier and Borodin instantiated for $A' = \textsc{Opt}(v)$ we get:

$$\sum_i u_i(b'_i(v), b_{-i}) \geq \sum_i \frac{v_i(\textsc{Opt}_i(v))}{2} - \sum_i \theta_i(\textsc{Opt}_i(v), b_{-i}) \geq \frac{1}{2} \sum_i v_i(\textsc{Opt}_i(v)) - c \sum_i p_i(b) \tag{7}$$

Now, by rearranging we get:

$$\sum_i u_i(b'_i(v), b_{-i}) + \sum_i p_i(b) \geq \sum_i \frac{v_i(\textsc{Opt}_i(v))}{2} - \sum_i \theta_i(\textsc{Opt}_i(v), b_{-i}) \geq \frac{1}{2} \sum_i v_i(\textsc{Opt}_i(v)) - (c-1) \sum_i p_i(b) \tag{8}$$

The latter states that our game satisfies our most relaxed semi-smoothness definition for $\lambda = \frac{1}{2}$ and $\mu = c - 1$. ■

In fact, we can improve our bound on the price of anarchy to $\frac{e}{e-1}c$ using the same trick of randomized deviations that we did in item bidding.

5 Variable Strategy Space Games and Universal Smoothness

In this section we cope with the following more general class of Bayesian Games whose goal is to capture games where the strategy space of a player not only his utility is dependend on his type. We denote with $A_i(t_i) \subseteq A_i$ the actions available to a player of type t_i. A players strategy is a function $s_i : T_i \to A_i$ that satisfies $\forall t_i \in T_i : s_i(t_i) \in A_i(t_i)$. We will denote with $A(t) = \times_i A_i(t_i)$. We still assume that the utility of a player depends on his type and the actions of the rest of the players: $u_i : T_i \times A \to \mathbb{R}$. We must point out that the utility of a player i is undefined if $a_i \notin A_i(t_i)$. The rest of the definitions are the same as in our previous definition of Bayesian Games. Our initial definition was a special case of this class of Bayesian Games where $\forall t_i \in T_i : A_i(t_i) = A_i$.

For such a Bayesian Game we need a slight alteration of the definition of what it means for a complete information instance of it to be smooth, since the utility of the complete information instance is undefined on strategies that are not in the strategy space of a player for that instance.

Definition 9 *A Bayesian Game is (λ, μ)-smooth if $\forall t \in T$ and for all $a, a' \in A(t)$:*

$$\sum_i u_i^{t_i}(a'_i, a_{-i}) \geq \lambda \sum_i u_i^{t_i}(a') - \mu \sum_i u_i^{t_i}(a)$$

The above class of games is a very general class of Bayesian Game and it is hard to believe that one can generalize smoothness to such a class. However, a lot of the games in the literature satisfy an even stronger definition of smoothness. For games that satisfy this stronger definition of smoothness we can generalize existing results to incomplete information versions of the games.

Definition 10 *A Bayesian Game is universally (λ, μ)-smooth iff $\forall t, w \in T$ and for all $a \in A(t)$, $b \in A(w)$:*

$$\sum_i u_i^{w_i}(b_i, a_{-i}) \geq \lambda \sum_i u_i^{w_i}(b) - \mu \sum_i u_i^{t_i}(a)$$

Since $b_i \in A_i(w_i)$ observe that the first term is also well defined.

Universal smoothness is a more restrictive notion than smoothness, in the sense that if a Bayesian Game is universally smooth then it is also smooth. This follows from the fact that if we take the definition of universal smoothness restricted only when $t = w$ then we get the smoothness definition. In addition the two definitions are equivalent to the smoothness of [11] for complete information games.

Though the above definition seems restrictive enough in the sections that follow we will show that most routing games studied in the literature are actually universally (λ, μ)-smooth and therefore the bounds known for the complete information carry over to some natural incomplete information versions.

Theorem 11 *If a Bayesian Game with Variable Strategy Space is universally (λ, μ)-smooth then it has Bayes-Nash PoA at most $(1 + \mu)/\lambda$.*

Proof. Using similar reasoning as in Theorem 2 we can arrive at the conclusion that:

$$\mathbb{E}_t[SW^t(s(t))] \;\geq\; \mathbb{E}_t\mathbb{E}_w[\sum_i u_i^{w_i}(\textsc{Opt}_i(w), s_{-i}(t_{-i}))]$$

Then we observe that $\textsc{Opt}(w) \in A(w)$ and $s(t) \in A(t)$. Thus applying the definition of universal smoothness we get:

$$\begin{aligned}\mathbb{E}_t[SW^t(s(t))] \;&\geq\; \mathbb{E}_t\mathbb{E}_w[\lambda \sum_i u_i^{w_i}(\textsc{Opt}(w)) - \mu \sum_i u_i^{t_i}(s(t))] \\ &=\; \lambda\mathbb{E}_w[SW^w(\textsc{Opt}(w))] - \mu\mathbb{E}_t[SW^t(s(t))]\end{aligned}$$

which gives the theorem. ∎

5.1 Weighted Congestion Games with Probabilistic Demands

In this section we examine how our analysis applies to incomplete information versions of routing games. The games that we study in this section are cost minimization games and hence we will use the variants of our theorems and notation so far adapted to cost minimization games.

We first describe the complete information game. We consider unsplittable atomic selfish routing games where each player has demand w_i that he needs to send from a node s_i to a node t_i over a graph G. Let $\mathcal{P}_i$ be the set of paths from s_i to t_i. The strategy of a player is to choose a path $p_i \in \mathcal{P}_i$. Each edge e of the graph has some delay function $l_e(x_e)$ where x_e denotes the total congestion of edge e: $x_e = \sum_{i:e\in p_i} w_i$, which is assumed to be monotone non-decreasing. Given a strategy profile $p = (p_i)_{i\in n}$ the cost of a player is $c_i(p) = \sum_{e\in p_i} w_i l_e(x_e)$.

In the literature so far only the case where w_i are common knowledge has been studied. This is a very strong informational assumption. Instead it is more natural consider the case where the w_i are private information and only a distribution on them is common knowledge. Thus the type a player in our game is his weight w_i. In fact to make the game comply with our definition of Bayesian Games we will assume that the strategy of a player is a pair (r_i, p_i) of a rate r_i and path p_i. In addition given a type w_i a player's action space is $A_i(w_i) = \{(w_i, p_i) : p_i \in \mathcal{P}_i\}$, i.e. we constraint the player to have to route his whole demand. Given the above small alteration in the definition of the game it is now easy to see that the cost of a player depends only on the strategies of the other players and not on their types: $\forall a_i = (r_i, p_i) \in A_i(t_i), \forall a_{-i} = (r_{-i}, p_{-i}) \in A_{-i} : c_i^{t_i}(a_i, a_{-i}) = \sum_{e\in p_i} r_i l_e(x_e(a))$ where $x_e(a) = \sum_{k:e\in p_k} r_k$ and it's undefined for $a_i \notin A_i(t_i)$. Hence, if we prove that the latter Bayesian Game is universally (λ, μ)-smooth, this would imply a Bayes-Nash PoA of $\lambda/(1-\mu)$.

Very recently (Bhawalkar et al [2]) showed that weighted congestion games are smooth games and therefore smoothness arguments provide tight results for the Price of Anarchy. Our analysis shows that one can extend these upper bounds to incomplete information too. Moreover, since complete information is a special case of incomplete information where priors are singleton distributions, the bayes-nash price of anarchy analysis will still be tight. Moreover, this shows a collapse of efficiency between complete and incomplete information and shows that knowing more doens't necessarily improve the efficiency guarrantee's in this types of games.

Most of the literature on weighted congestion games uses the following fact: if for the class of delay functions $\mathcal{C}$ allowed we have that: $\forall x, x^* \in \mathbb{R}^+ : x^* l_e(x + x^*) \geq \lambda x^* l_e(x^*) + \mu x l_e(x)$ then weighted congestion games with delays in class $\mathcal{C}$ are (λ, μ)-smooth. We will actually show that if the delay functions satisfy the above property then the Bayesian Game is universally (λ, μ)-smooth.

Lemma 12 *If for any delay function $l_e()$ in the class of delay functions $\mathcal{C}$ allowed we have that: $\forall x, x^* \in \mathbb{R}^+ : x^* l_e(x + x^*) \geq \lambda x^* l_e(x^*) + \mu x l_e(x)$ then the resulting class of Bayesian Unsplittable Selfish Routing Games with Probabilistic Demands is universally (λ, μ)-smooth.*

Proof. Let w, t be two type profiles. Let $a = (w, p) \in A(w)$ and $b = (t, p') \in A(t)$. Let $x_e(a) = \sum_{i:e\in p_i} w_i$ and $x_e(b) = \sum_{i:e\in p'_i} t_i$. Then:

$$\begin{aligned}
\sum_i c_i^{t_i}(b_i, a_{-i}) \leq& \sum_i \sum_{e\in p'_i} t_i l_e(t_i + x_e(a)) \leq \sum_i \sum_{e\in p'_i} t_i l_e(x_e(b) + x_e(a)) \\
=& \sum_e x_e(b) l_e(x_e(b) + x_e(a)) \\
\leq& \lambda \sum_e x_e(b) l_e(x_e(b)) + \mu \sum_e x_e(a) l_e(x_e(a)) \\
=& \lambda \sum_i \sum_{e\in p'_i} t_i l_e(x_e(b)) + \mu \sum_e \sum_{e\in p_i} w_i l_e(x_e(a)) \\
=& \lambda \sum_i c_i^{t_i}(b) + \mu \sum_i c_i^{w_i}(a)
\end{aligned}$$

which is exactly the universal smoothness definition. ■

5.2 Bayesian Effort Games

In this section we study what our analysis imply for incomplete information versions of the following class of effort games [1]: There is a set of players $[n]$ and a set of project $[m]$. Each player has some budget of effort B_i which he can split among the projects. Each project j has some value that is a non-decreasing concave function $V_j()$ of the weighted sum of efforts $\sum_{i\in N} a_{ij}x_{ij}$ where a_{ij} is some ability factor of player i in project j (we assume $V(0) = 0$). The value of a project is then split among the participants and each participant receives a share proportional to his weighted input $a_{ij}x_{ij}$. Such games where shown in [1] to be Valid Utility Games [13] and hence $(1, 1)$-smooth.

We will consider the natural incomplete information version of these games where each players ability vector $a_i = (a_{ij})_{j\in[m]}$ and the budget B_i are private information, each ability vector is drawn from some distribution F_i on $\mathbb{R}_+^{m+1}$. The F_i are independent. To adapt it in our variable strategy space model we will assume that the strategy of a player is a pair $(\tilde{a}_i, x_i)$ where $\tilde{a}_i$ is the declared ability vector and x_i is the vector of efforts of player i. We will constraint the strategy space such that given an ability vector a_i the player has to declare his true ability vector: $A_i(a_i, B_i) = \{(a_i, x_i) : x_i \in \mathbb{R}^m, \sum_j x_{ij} \leq B_i\}$.

We are able to show that these games are actually universally smooth games and thereby the Bayes-Nash PoA of the above Bayesian Games will be at most 2.

Lemma 13 *Bayesian Effort Market Games are universally* $(1, 1)$*-smooth.*

Proof. The proof is an adaptation of the smoothness proof for valid utility games [13, 11], but in the space of real functions instead of set functions. In addition it is adapted to accommodate for different types of players so as to show the stronger version of universal smoothness is satisfied.

Let $s = (a, x) \in A(a, B)$ and $s' = (b, y) \in A(b, B')$. Then we have:

$$\sum_i u_i^{(b_i,B_i)}(s'_i, s_{-i}) = \sum_i \sum_j b_{ij}y_{ij} \frac{V_j(b_{ij}y_{ij} + a_{-i} \cdot x_{-i})}{b_{ij}y_{ij} + a_{-i} \cdot x_{-i}}$$

Now we use the fact that for a concave function $V_j()$ that satisfies $V_j(0) = 0$, it holds that $V_j(x)/x$ is a decreasing function. Hence:

$$\frac{V_j(b_{ij}y_{ij} + a_{-i} \cdot x_{-i})}{b_{ij}y_{ij} + a_{-i} \cdot x_{-i}} \leq \frac{V_j(a_{-i} \cdot x_{-i})}{a_{-i} \cdot x_{-i}} \implies b_{ij}y_{ij} \frac{V_j(b_{ij}y_{ij} + a_{-i} \cdot x_{-i})}{b_{ij}y_{ij} + a_{-i} \cdot x_{-i}} \geq V_j(b_{ij}y_{ij} + a_{-i} \cdot x_{-i}) - V_j(a_{-i} \cdot x_{-i})$$

Thus:

$$\sum_i u_i^{(b_i,B_i)}(s'_i, s_{-i}) \geq \sum_i \sum_j (V_j(b_{ij}y_{ij} + a_{-i} \cdot x_{-i}) - V_j(a_{-i} \cdot x_{-i}))$$

In addition, since $V_j()$ is concave, increasing then for all $t_1 > t_2$ and $y > 0$: $V_j(y + t_1) - V_j(t_1) \leq V_j(y + t_2) - V_j(t_2)$. Combining we get:

$$\begin{aligned}
\sum_i u_i^{(b_i,B_i)}(s'_i, s_{-i}) \geq & \sum_j \sum_i (V_j(b_{ij}y_{ij} + a_{-i} \cdot x_{-i}) - V_j(a_{-i} \cdot x_{-i})) \\
\geq & \sum_j \sum_i V_j\left(b_{ij}y_{ij} + a_{-i} \cdot x_{-i} + a_{ij}x_{ij} + \sum_{k=1}^{i-1} b_{kj}y_{kj}\right) - V_j\left(a_{-i} \cdot x_{-i} + a_{ij}x_{ij} + \sum_{k=1}^{i-1} b_{kj}y_{kj}\right) \\
= & \sum_j \sum_i V_j\left(\sum_{k=1}^{i}(b_{kj}y_{kj} + a_{kj}x_{kj}) + \sum_{k=i+1}^{n} a_{kj}y_{kj}\right) - V_j\left(\sum_{k=1}^{i-1}(b_{kj}y_{kj} + a_{kj}x_{kj}) + \sum_{k=i}^{n} a_{kj}y_{kj}\right) \\
= & \sum_j V_j\left(\sum_{k=1}^{n}(b_{kj}y_{kj} + a_{kj}x_{kj})\right) - V_j\left(\sum_{k=1}^{n} a_{kj}x_{kj}\right) \\
\geq & \sum_j V_j\left(\sum_{k=1}^{n} b_{kj}y_{kj}\right) - V_j\left(\sum_{k=1}^{n} a_{kj}x_{kj}\right) \\
= & SW^{b,B'}(s') - SW^{a,B}(s)
\end{aligned}$$

■

Corollary 14 *The Bayes-Nash PoA of Bayesian Effort Games is at most* 2.

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